

On the Transition from Superconducting to Normal Phase, Accounting for Latent Heat and Eddy Currents

Abstract: A rigorous solution is given for the superconducting transition of a semi-infinite slab held at a point below the critical temperature T_c when a constant magnetic field above the critical value H_{oc} is applied. The solution accounts for both the absorption of latent heat and the dissipation of eddy-current heat during the transition. A numerical example is calculated for the case of constants close to those of tantalum.

Introduction

It is well known that the transition between the superconducting and normal phase of metals is affected by electromagnetic damping according to Maxwell's equations. A rigorous solution for the switching of a plane half-space, (Fig. 1), in which the critical field is assumed to be independent of temperature, has been obtained by Pippard.¹ For a similar problem, the propagation of a phase boundary during the melting or solidification of a material, a rigorous solution has been given by Carslaw and Jaeger.² Although Pippard and Carslaw-Jaeger dealt with essentially the same mathematical problem, their methods of solution were different.

Measurements on tantalum, however, have shown that it is not always permissible to consider the material as isothermal, in view of both the absorption of latent heat and the production of eddy-current heat. An approximate solution has been obtained by Ittner,³ in which he considers an "effective latent heat," which assumes a hypothetical dissipation of Joule heat from the moving boundary.

The results for a certain sample of tantalum according to Pippard's theory and Ittner's results are given in Fig. 2. This paper is concerned with a rigorous solution which accounts for both eddy-current heating and latent heat. Calculations have been carried out for values of parameters which are typical for actual specimens; these calculations are expected to be close to Ittner's values.

The rigorous theory given here derives a reciprocal switching time, of which the second derivative with respect to the applied magnetic field varies monotonically with the field. We find the same monotony in Ittner's experimental results. Ittner's theoretical result, however, shows a non-

monotony in this second derivative. Since the physical constants in Ittner's sample were not exactly known, one might attempt to define the basic constants by fitting the measured values to the calculated curves. This method of determining the constants does not appear to be justified, however, since the measurements were made on a hollow cylinder instead of a semi-infinite slab, and since there are uncertain effects due to the surface energies and to possible thermal barriers at the outer boundary.

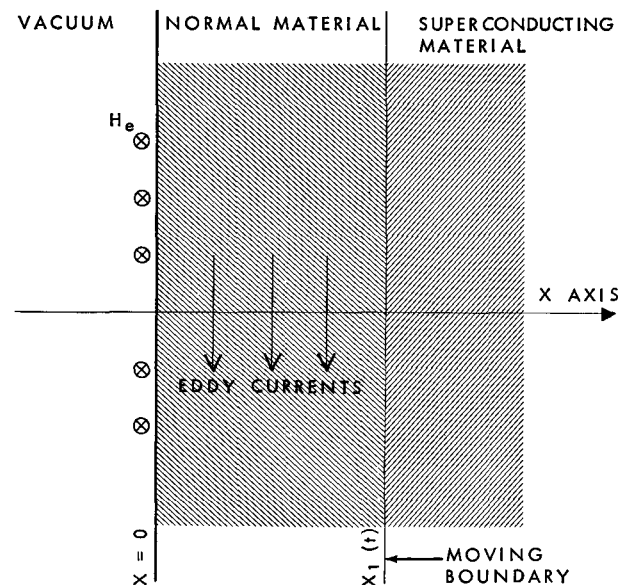


Figure 1 Switching of a plane half-space in a semi-infinite solid.

*Philips Research Laboratories, Eindhoven, Netherlands. The work reported here was done at the IBM Research Center, Poughkeepsie, N. Y.

In order to check the model used in this paper, future measurements should be made on specimens for which the basic physical constants are precisely known.

Mathematical formulation

A semi-infinite solid, which coincides with the half-space $x \geq 0$ is superconducting for $t < 0$, as indicated in Fig. 1. The temperature T of the solid then equals T_c , which is smaller than the critical temperature T_c . No magnetic field is present for $t < 0$. At the time $t = 0$ a homogeneous magnetic field of constant strength H_0 , greater than the critical field H_{0c} at $T = T_c$ is applied perpendicular to the x -axis.

After a certain time $t > 0$ the semi-infinite solid consists of a normal conducting region $0 < x < x_1(t)$ and a superconducting region $x > x_1(t)$. The separation between the normal and the superconducting regions is a function $x_1(t)$ of time and will be called the moving boundary.

The magnetic field $H = H(x, t)$ in the normal conducting region satisfies the equation:

$$\frac{\partial^2 H}{\partial x^2} = \frac{1}{k} \frac{\partial H}{\partial t}, \quad 0 < x < x_1(t), \quad (1)$$

where $k = c^2/4\pi\sigma$ and c is the velocity of light in *vacuo* and σ the conductivity of the solid.

The boundary conditions for the magnetic field are:

$$H(0, t) = H_0, \quad t > 0 \quad (2)$$

and

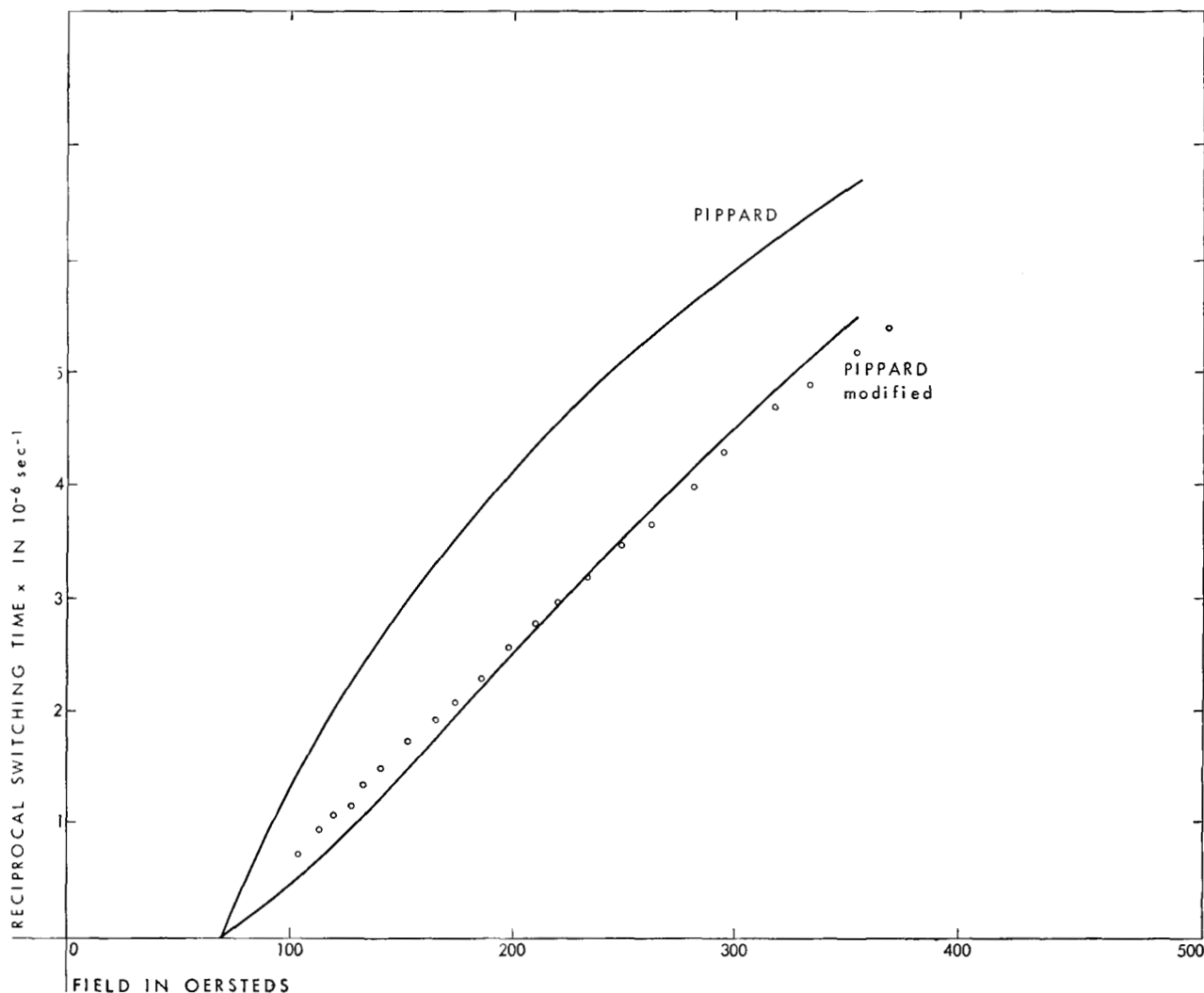
$$H[x_1(t), t] = H_c(T_1) = H_c, \quad t > 0, \quad (3)$$

where T_1 is the temperature at the moving boundary and $H_c(T)$ is a given function of the temperature T .

Furthermore the derivative of the magnetic field with respect to x in the normal material at the moving boundary is related to the velocity of the moving boundary in the following way:

$$\left(\frac{\partial H(x, t)}{\partial x} \right)_{x=x_1(t)} = -\frac{H_c}{k} \frac{dx_1(t)}{dt}. \quad (4)$$

Figure 2 Previous data for switching time of tantalum. Calculations by Pippard for semi-infinite plane geometry; Pippard's calculations modified by Iltner to include approximate thermal effects.



The transition from the superconducting to the normal state is accompanied by the absorption of latent heat and the dissipation of heat due to the eddy currents.

The temperature in the normal material T_N and the temperature T_S in the superconducting material satisfy the equations (5) and (6).

$$\frac{\partial^2 T_N}{\partial x^2} = \frac{1}{\kappa_N} \frac{\partial T_N}{\partial t} - \frac{1}{\sigma K_N} j^2, \quad 0 < x < x_1(t). \quad (5)$$

$$\frac{\partial^2 T_S}{\partial x^2} = \frac{1}{\kappa_S} \frac{\partial T_S}{\partial t}, \quad x > x_1(t), \quad (6)$$

where the eddy current j is given by:

$$j = -\frac{c}{4\pi} \frac{\partial H}{\partial x}. \quad (7)$$

The constants κ_N and κ_S are the diffusivities of the normal and superconducting material, respectively, and K_N denotes

the thermal conductivity of the normal material.

The temperature must be continuous at the moving boundary while the difference of the derivatives of the temperatures with respect to x at the moving boundary equals the amount of latent heat absorbed at the moving boundary.

We therefore have:

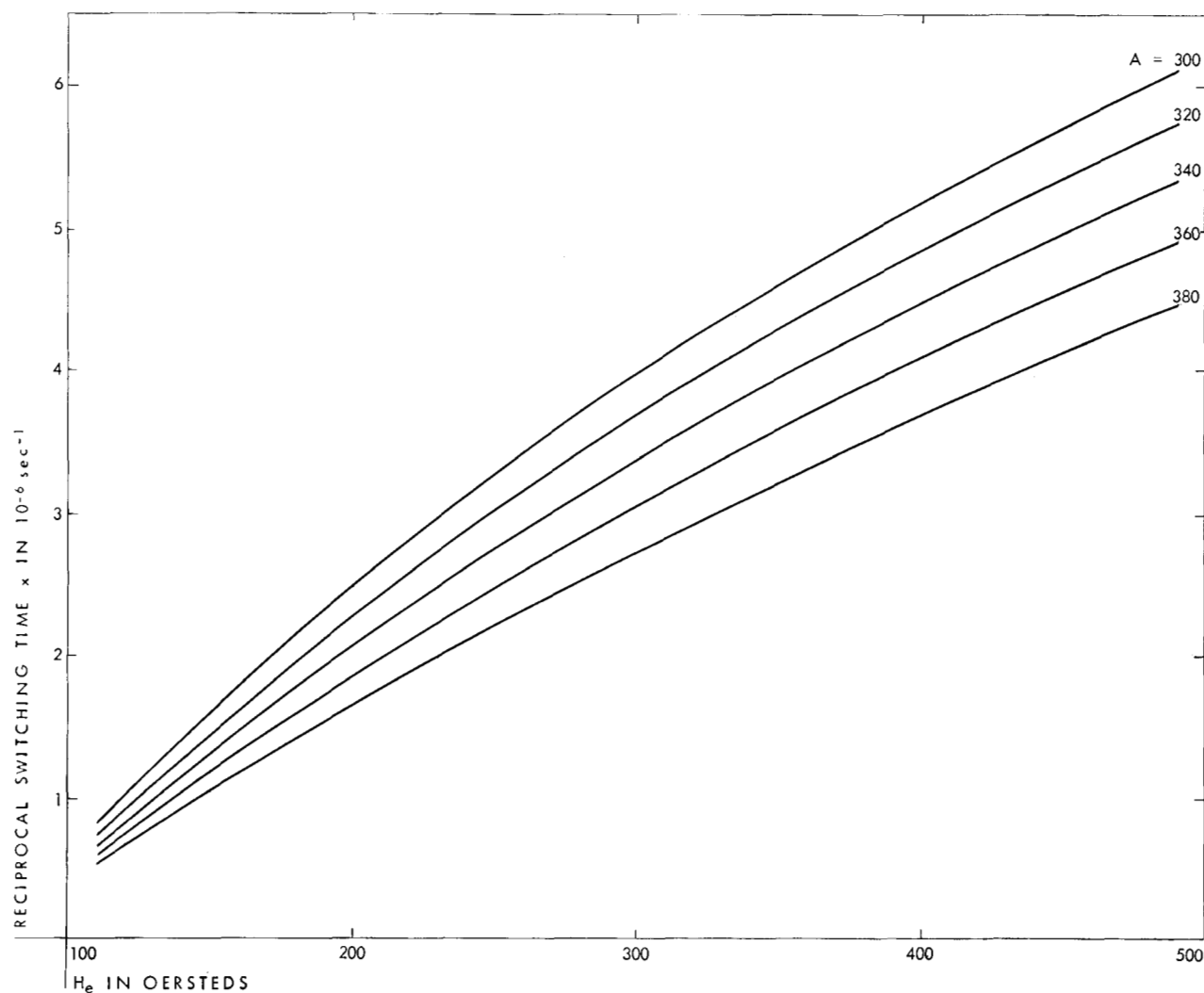
$$T_N[x_1(t), t] = T_S[x_1(t), t], \quad t > 0 \quad (8)$$

and

$$K_N \left(\frac{\partial T_N}{\partial x} \right)_{x=x_1(t)} = K_S \left(\frac{\partial T_S}{\partial x} \right)_{x=x_1(t)} - L(T) \frac{dx_1(t)}{dt}, \quad t > 0, \quad (9)$$

where K_S denotes the thermal conductivity of the superconducting material. The function $L(T)$ is the latent heat and is a given function of T .

Figure 3 Reciprocal switching time as a function of magnetic driving field H_e for thermal conductivity $K_N = 0.05 \times 10^7$ erg/(cm sec °K). The slopes of the $H-T$ curve: $A = 300(20)380$.



Finally, there are two more boundary conditions for the temperatures T_N and T_S :

$$T_N(0,t) = T_c, \quad t > 0 \quad (10)$$

and

$$T_S(\infty,t) = T_c, \quad 0 < t < \infty. \quad (11)$$

The initial conditions of the problem are:

$$x(0) = 0 \quad (12)$$

and

$$T_S(x,0) = T_c, \quad x > 0. \quad (13)$$

Analytical solution of the problem

We try to find a solution of the equations which is a function of $u = xt^{-1/2}$ only. Apparently we have

$$\frac{\partial}{\partial x} = \frac{1}{t^{1/2}} \frac{d}{du},$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{t} \frac{d^2}{du^2},$$

and

$$\frac{\partial}{\partial t} = -\frac{u}{2t} \frac{d}{du}.$$

Using this transformation and inserting the operators into (1), (2), (3), (6), (8), (10) and (11) we find:

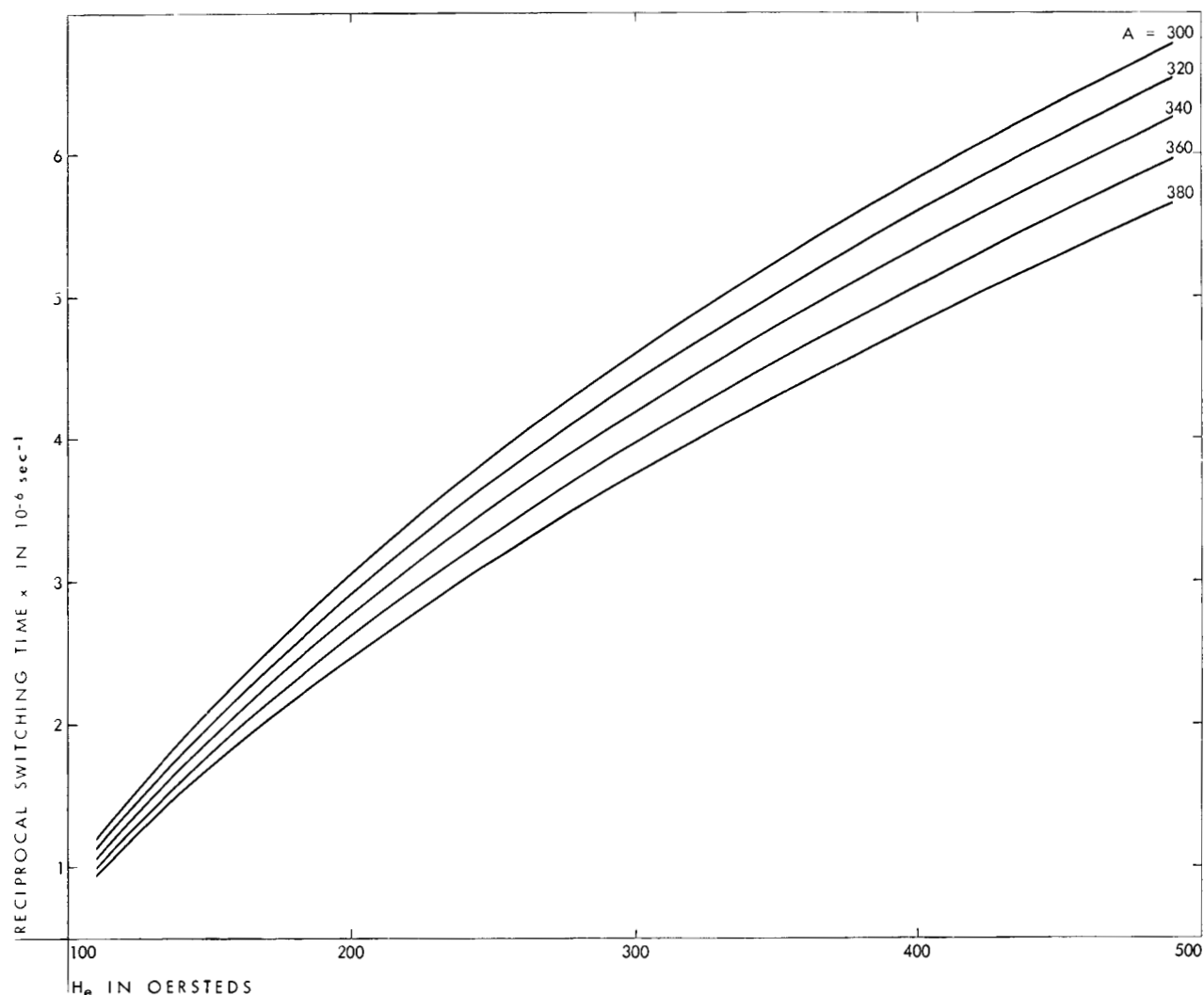
$$\frac{d^2 H}{du^2} + \frac{u}{2k} \frac{dH}{du} = 0 \quad (1a)$$

$$H(0) = H_c \quad (2a)$$

$$H(u_1) = H_c, \quad (3a)$$

$$\text{with } u_1 = x_1(t)t^{-1/2}.$$

Figure 4 Reciprocal switching time as a function of magnetic driving field H_0 for thermal conductivity $K_N = 0.15 \times 10^7$ erg/(cm sec °K). The slopes of the $H-T$ curve: $A = 300(20)380$.



$$\frac{d^2 T_S}{du^2} + \frac{u}{2K_S} \frac{\partial T_S}{\partial u} = 0$$

$$T_N(u_1) = T_S(u_1)$$

$$T_N(0) = T_e$$

$$\text{and } T_S(\infty) = T_e$$

The transformation $u = \chi t^{-\frac{1}{2}}$ in (7) yields:

$$j = -\frac{c}{4\pi t^{\frac{1}{2}}} \frac{dH}{du}$$

Consequently (5) transforms into the following equation:

$$\frac{d^2 T_N}{du^2} = -\frac{u}{2K_N} \frac{dT_N}{du} - \frac{c^2}{16\pi^2 \sigma K_N} \left(\frac{dH}{du} \right)^2 \quad (5a)$$

Equations (4) and (9) finally become

$$(6a)$$

$$\left(\frac{dH}{du} \right)_{u=u_1} = -\frac{1}{k} H_c(T_1) \left(\frac{u_1}{2} + t \frac{du_1}{dt} \right) \quad (4a)$$

$$(8a)$$

and

$$(10a)$$

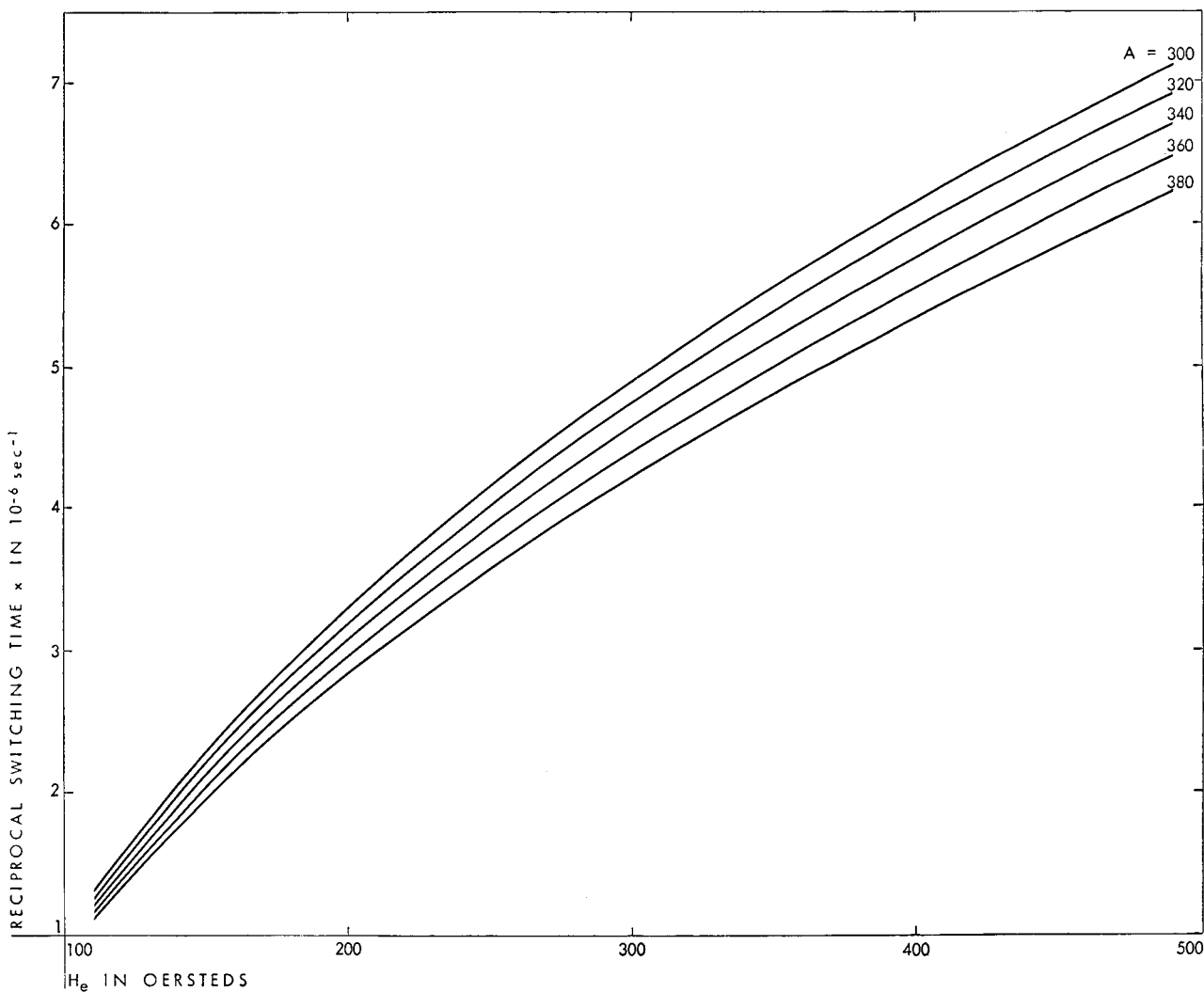
$$(11a)$$

$$K_N \left(\frac{dT_N}{du} \right)_{u=u_1} = K_S \left(\frac{dT_S}{du} \right)_{u=u_1} - L(T_1) \left(\frac{u_1}{2} + t \frac{du_1}{dt} \right) \quad (9a)$$

Closer investigation of (4a) and (9a) shows that all the equations become independent of t if we make the choice $u_1 = \alpha k^{\frac{1}{2}}$, where α is a constant. This implies that the temperature T_1 and critical field $H_c(T_1)$ at the moving boundary are constants. Furthermore the latent heat $L(T_1)$ at the moving boundary does not depend on time.

It remains to obtain solutions of (1a), (5a) and (6a) which satisfy the corresponding boundary conditions. A solution

Figure 5 Reciprocal switching time as a function of magnetic driving field H_e for thermal conductivity $K_N = 0.25 \times 10^7$ erg/(cm sec °K). The slopes of the $H-T$ curve: $A = 300(20)380$.



of (1a) and (2a) is apparently

$$H = H_e - \beta \operatorname{erf} \frac{u}{2k^{\frac{1}{2}}}, \quad (14)$$

where

$$\operatorname{erf} w = \frac{2}{\sqrt{\pi}} \int_0^w e^{-\xi^2} d\xi$$

and β is a constant.

A similar solution can be obtained for (6a) and (11a),

$$T_s = T_e - \delta \operatorname{erfc} \frac{u}{2\kappa_s^{\frac{1}{2}}}, \quad (15)$$

with $\operatorname{erf} w = 1 - \operatorname{erfc} w$ and δ a constant.

We notice that the initial condition (13) is also satisfied by (15). Finally, it can be verified that:

$$T_N = T_e - \gamma \operatorname{erf} \frac{u}{2\kappa_N^{\frac{1}{2}}} + \Lambda \int_0^u \exp\left(-\frac{t^2}{4\kappa_N}\right) dt \int_0^t \exp\left[-\xi^2\left(\frac{1}{2k} - \frac{1}{4\kappa_N}\right)\right] d\xi, \quad (16)$$

with

$$\Lambda = -\frac{c^2\beta^2}{16\pi^3\sigma K_N k} = -\frac{\beta^2}{4\pi^2 K_N}$$

and γ a constant, is a solution of (5a) and (10a).

The constants α , β , γ and δ can be defined by the conditions (3a), (4a), (8a) and (9a).

After putting $\frac{H_e - H_c}{H_c} = p$ and using the expressions (14), (15) and (16) in these conditions one finds, after some calculations,

$$\alpha \sqrt{\pi} \exp\left(\frac{\alpha^2}{4}\right) \operatorname{erf} \frac{\alpha}{2} = 2p \quad (17)$$

$$\beta = \frac{H_e - H_c}{\operatorname{erf} \frac{\alpha}{2}} \quad (18)$$

$$\Lambda \int_0^{\alpha k^{\frac{1}{2}}} \exp\left(-\frac{u^2}{4\kappa_N}\right) du \int_0^u \exp\left[-\xi^2\left(\frac{1}{2k} - \frac{1}{4\kappa_N}\right)\right] d\xi - \gamma \operatorname{erf} \frac{\alpha}{2} \left(\frac{k}{\kappa_N}\right)^{\frac{1}{2}} = -\delta \operatorname{erfc} \frac{\alpha}{2} \left(\frac{k}{\kappa_s}\right)^{\frac{1}{2}} \quad (19)$$

$$\Lambda K_N \exp\left(-\frac{\alpha^2 k}{4\kappa_N}\right) \int_0^{\alpha k^{\frac{1}{2}}} \exp\left[-\xi^2\left(\frac{1}{2k} - \frac{1}{4\kappa_N}\right)\right] d\xi - \gamma \frac{K_N \exp\left(-\frac{\alpha^2 k}{4\kappa_N}\right)}{\sqrt{\pi \kappa_N}} = \delta \frac{K_s \exp\left(-\frac{\alpha^2 k}{4\kappa_s}\right)}{\sqrt{\pi \kappa_s}} - L(T_1) \frac{\alpha k^{\frac{1}{2}}}{2}, \quad (20)$$

$$\text{with } \Lambda = -\frac{\beta^2}{4\pi^2 K_N}$$

and finally

$$\psi(T_1) = T_1 - T_e + \delta \operatorname{erfc} \frac{\alpha}{2} \left(\frac{k}{\kappa_s}\right)^{\frac{1}{2}} = 0 \quad (21)$$

We notice that the solution for the moving boundary $x_1(t) = \alpha(k t)^{\frac{1}{2}}$ also satisfies condition (12). If we now let the boundary move a distance Δx , the switching time t_s will be

$$t_s = \frac{(\Delta x)^2}{\alpha^2 k}.$$

Numerical solution

We observe that for a given field H_e and temperature T_1 , α can be obtained from (17). Furthermore γ and δ can be derived from two linear equations (19) and (20). Finally T_1 can be changed until (21) is fulfilled.

In order to find α from (17), we used Newton's method for finding the root of the function $\varphi(\alpha)$:

$$\varphi(\alpha) = \alpha \sqrt{\pi} \exp\left(\frac{\alpha^2}{4}\right) \operatorname{erf} \frac{\alpha}{2} - 2p. \quad (22)$$

The value of the temperature T_1 was found as follows. We started with an arbitrary value $T_1 = T_{10}$ and computed the function

$$\psi(T_{10} + kh), \quad k = 1, 2, 3, \dots$$

with h a given constant. As soon as $\{\psi(T_{10} + kh)\} \cdot \{\psi(T_{10} + (k+1)h)\} < 0$ was found, *regula falsi* was used to obtain the zero of $\psi(T_1)$.

For the computations of the integrals occurring in (19) and (20) their Taylor expansions were used. These expansions are as follows:

$$\int_0^{\alpha k^{\frac{1}{2}}} \exp\left(-\frac{u^2}{4\kappa_N}\right) du \int_0^u \exp\left[-\xi^2\left(\frac{1}{2k} - \frac{1}{4\kappa_N}\right)\right] d\xi = k \sum_{s=0}^{\infty} b_s \frac{\alpha^{2s+2}}{2s+2} \quad (23)$$

and

$$\exp\left(-\frac{\alpha^2 k}{4\kappa_N}\right) \int_0^{\alpha k^{\frac{1}{2}}} \exp\left[-\xi^2\left(\frac{1}{2k} - \frac{1}{4\kappa_N}\right)\right] d\xi = k^{\frac{1}{2}} \sum_{s=0}^{\infty} b_s \alpha^{2s+1}, \quad (24)$$

$$\text{with } b_s = \frac{(-)^s \frac{1}{2^s s!} - \frac{k}{2\kappa_N} b_{s-1}}{2s+1} \text{ and } b_0 = 1.$$

Calculation of a numerical example

As an example we choose the data for the constants in the problem close to those of tantalum. The reciprocal switching time t_s^{-1} is drawn in Figs. 3 through 5 and the temperature T_1 at the moving boundary is given in Figs. 6 through 8. Numerical data are available from the author.

The following choices were made:

$$\kappa_s = \frac{k_s}{6.44 \times 10^4 \text{ sec}}, \quad \kappa_N = \frac{K_N}{3.22 \times 10^4 \text{ sec}}$$

$$\text{with } K_N = K_s = 0.05 \times 10^7 (0.05 \times 10^7) 0.25 \times 10^7 \frac{\text{erg}}{\text{cm sec } ^\circ \text{K}}.$$

The data selected for Figs. 3 to 8 were in the range (0.10×10^7) .

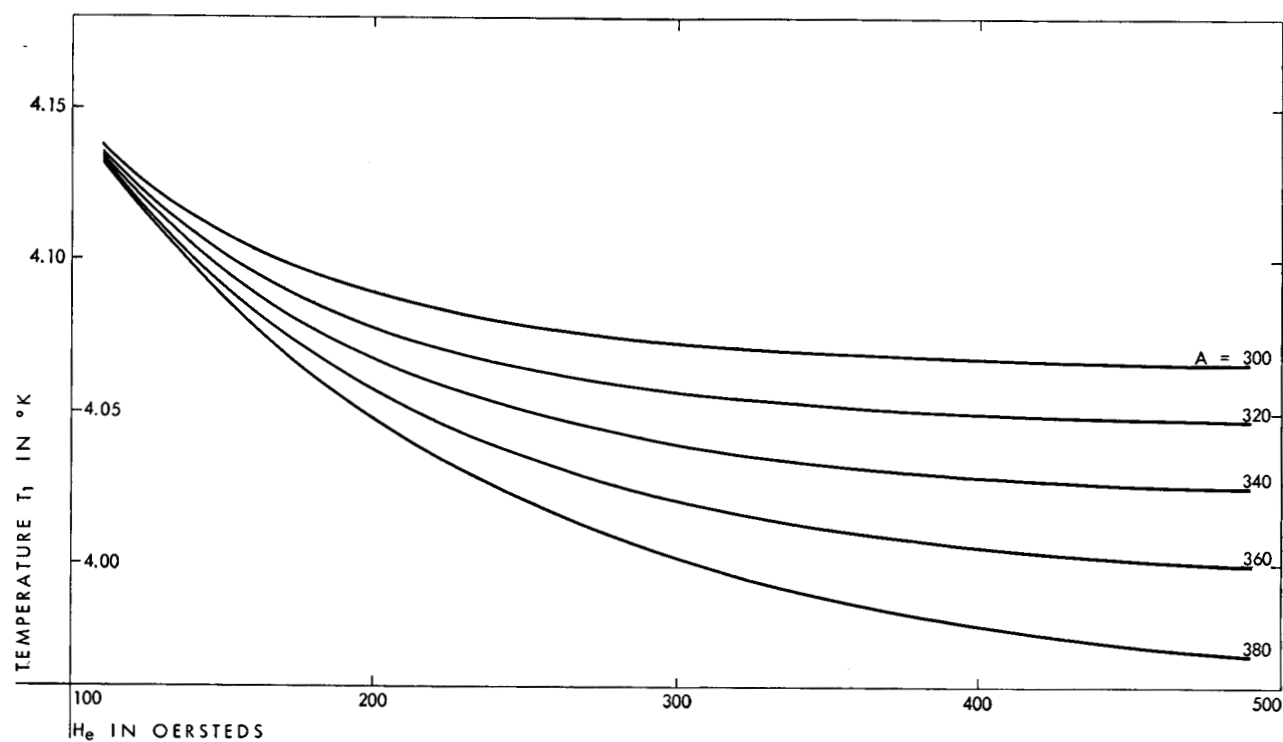


Figure 6 $K_N = 0.05 \times 10^7$ erg/(cm sec °K).

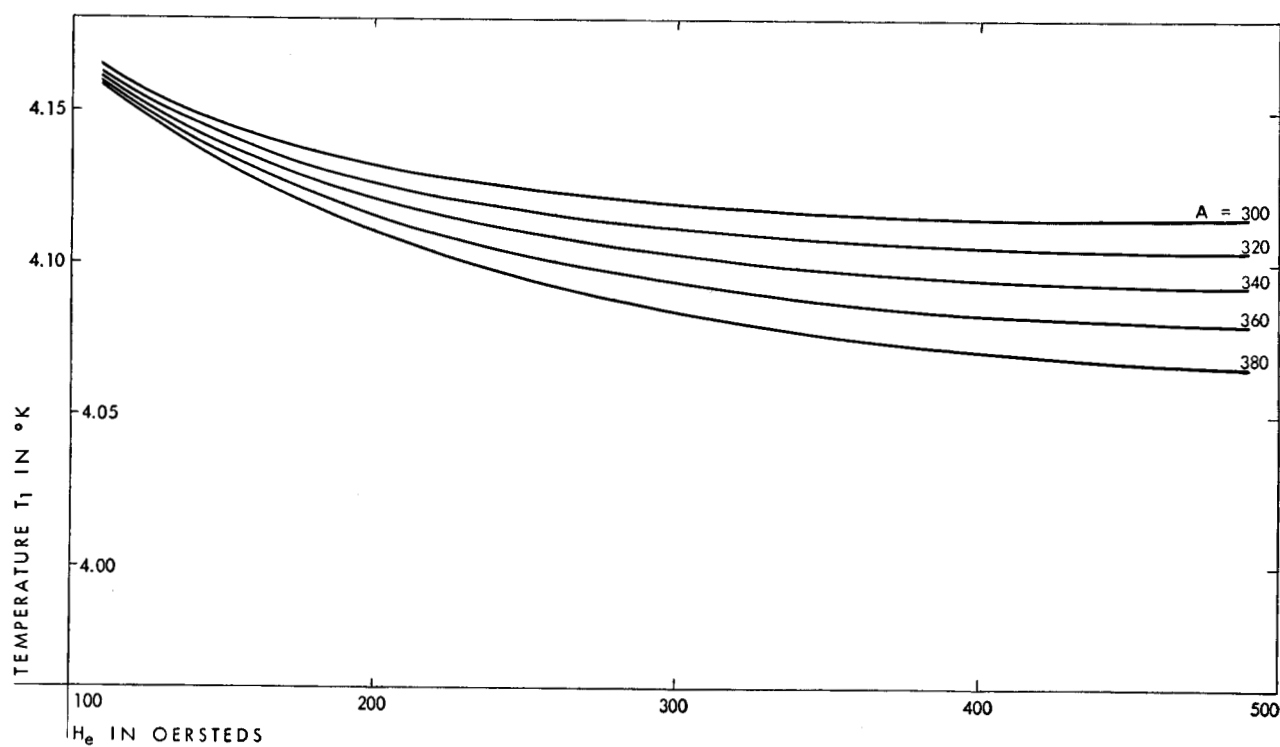


Figure 7 $K_N = 0.15 \times 10^7$ erg/(cm sec °K).

Figures 6, 7 and 8 Temperature at the moving boundary vs driving magnetic field for given values of K_N and A .

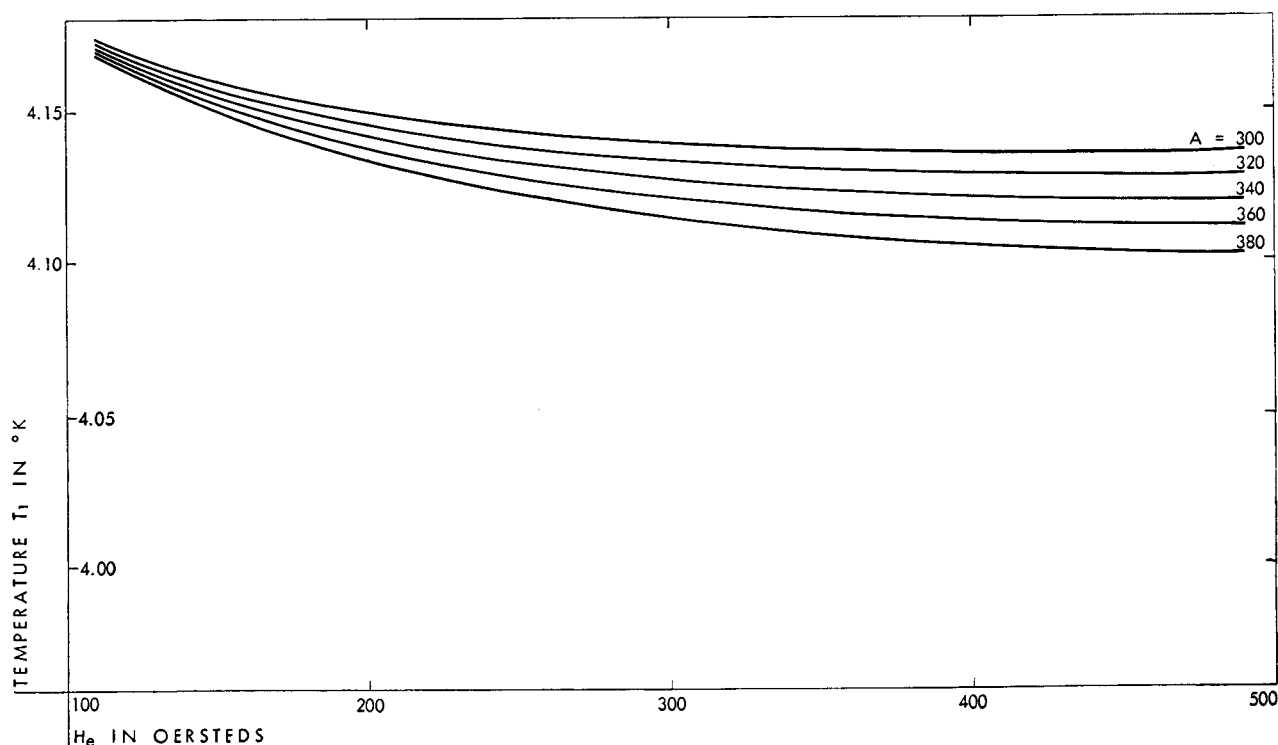


Figure 8 $K_N = 0.25 \times 10^7$ erg/(cm sec °K).

For the critical field H_c we choose:

$$H_c = H_{oc} - A(T - T_e).$$

Consequently the latent heat becomes

$$L(T) = \frac{TAH_c}{4\pi}.$$

The temperature T_e of the bath was chosen to be equal to 4.2°K and the field $H_{oc} = 68$ oersteds. The conductivity σ had the value $1/(0.48 \mu\text{ohm-cm})$.

As indicated in the Figures, the values for A were taken in the range $A = 300(20)380$.

We calculated H_c , T_1 , α , β , γ , δ and t_s as a function of H_e up to 4 decimal places. The travel distance Δx was chosen as 0.0048 cm.

A program was written by the author for the IBM 704 digital computer in order to carry out the necessary calculations. In the program one can choose K_N , K_S , σ , A and H_{oc} arbitrarily. Furthermore the range in which H varies can arbitrarily be chosen.

Acknowledgment

I am indebted to Dr. J. Swihart for drawing my attention to the problem and for the useful discussions we had about the problem, and to Dr. A. Mitchell and Dr. W. B. Ittner, for their valuable remarks. I wish to express my appreciation to IBM for having made it possible to do this work. Furthermore I would like to acknowledge the use of the IBM 704 at the Computing Center of the IBM Research Laboratory.

List of symbols

- A = negative slope of $H-T$ curve at $T = T_e$
- c = velocity of light in vacuo
- Δx = travel distance of moving boundary
- H = magnetic field
- H_c = critical field
- H_e = magnetic field at $x=0$
- H_{oc} = critical field at $T = T_e$
- j = eddy current
- K_N = thermal conductivity of normal material
- K_S = thermal conductivity of superconducting material
- κ_N = diffusivity of the normal material
- κ_S = diffusivity of the superconducting material
- σ = electrical conductivity of the solid
- t = time
- t_s = switching time
- T_1 = temperature at the moving boundary
- T_e = temperature of the bath, i.e., at $x=0$ and $x=\infty$
- T_N = temperature of normal material
- T_S = temperature of superconducting material
- x = distance
- $x_1(t)$ = distance of the moving boundary

References

1. A. B. Pippard, "Kinetics of the Phase Transition in Superconductors," *Phil. Mag.* **41**, 243 (1950).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, p. 71 ff. (1947).
3. W. B. Ittner, "The Superconducting-to-Normal Phase Transition in Tantalum," *Phys. Rev.* **111**, 1483 (Sept. 15, 1958).

Received June, 1958